MATHEMATICAL VIRUSES*

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ABSTRACT. The discovery of Mathematical Viruses is announced here for the first time. Such viruses are a serious threat to the general mental health of the mathematical community. Several viruses inimical to the unity of mathematics are identified, and their deleterious characteristics are described. A strong dose of geometric algebra and calculus is the best medicine for both prevention and cure.

1. INTRODUCTION

Computer viruses have been prominent in the news lately. The increasingly widespread and frequent communication among computers has facilitated the spread of computer viruses to the point where viruses are seriously regarded as a threat to national security in the United States.

The computer virus (CV) owes its name and perhaps its genesis to the biological virus (BV). Like a BV, a CV cannot function by itself, but when attached to a host it replicates repeatedly until it impairs the functioning of the host, sometimes to the point of disabling the host altogether. Moreover, it is infectious, spreading from one host to another on contact. The host of a CV is a computer program, to which the CV is attached as a subroutine, replicating and spreading whenever the host program is run. Similarly, a BV is a fragment of DNA containing instructions for self-replication which are activated when the BV is in a living cell. Fortunately, antiviral agents can be developed to cure infected computers as well as biological organisms.

My purpose here is to call your attention to another kind of virus — one which can infect the mind — the mind of anyone doing mathematics, from young student to professional mathematician. As I believe this is the first published paper to explicitly identify such viruses, I take the liberty of naming and describing them as follows: A mathematical virus (MV) is a preconception about the structure, function or method of mathematics which impairs one's ability to do mathematics. Just as a CV is program which impairs the operating system of a computer, an MV is an idea which impairs the conceptualization of mathematics in the mind. Indeed, as one definition of "virus," Webster gives "something

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that corrupts the mind or soul." Since the identification and classificalion of MVs has only just begun, it would be premature to attempt a more precise definition. The better course is to examine some specific viruses to form a firm empirical base for further study.

My first example is an easily recognized MV which is extremely virulent and as common as the common cold. I call it the *coordinate virus*, designating it by MV/C to denote genesis and type, and characterizing it as follows:

MV/C: Coordinates are essential to calculations.

Physicists and engineers are especially susceptible to this virus, because most of their textbooks are infected, and infected teachers pass it on to their students. Mathematicians as a group are less susceptible, because many of them have been innoculated with a good course in abstract algebra, though, as we shall see, some resistant strains have survived in certain mathematical subspecialties.

The diagnosis and treatment of MV infections is still in its infancy, and it is especially delicate when the infected host is not aware of his illness, as is usually the case. Diagnosis of infection with the coordinate virus MV/C is comparatively easy, owing to the presence of a well-defined syndrome of symptoms which I call *coordinitis:* Typically, the infected subject fails to distinguish the abstract vector concept from its matrix representation and consequently has great difficulty conceiving and manipulating invariant functional relations among vectors without expressing them in terms of coordinates; he is likely to regard the real and complex numbers as more fundamental, or even "more real" than vectors. To cite a specific example of such symptoms, one textbook by a distinguished physicist asserts that "the vector calculus is like a folding ruler, before you can use it you have to unfold it" (by which he meant, decompose vector formulas into components). In my own experience of more than two decades teaching physics graduate students, I have observed that most of them suffer from coordinitis and as many as 25% may be permanently crippled by the disease.

Some may think that the coordinate virus is harmless or benign. After all, it is not fatal. The afflicted can still limp along in their mathematical thinking. However, they are condemned to a world of prosaic mathematical applications. They will never be able to scale the Olympic heights to inhale the pristine air of abstract mathematics. Let there be no mistake about the nature of the coordinate virus. There is nothing wrong with using coordinates when they are appropriate. It is the insidious idea that coordinates are somehow more fundamental or concrete than other mathematical objects that limit conceptual capacity.

Though I would like to lay claim to the important discovery that mathematical viruses exist, honesty compels me to admit that mathematicians must have known about them all along. For as soon as a mathematician is introduced to the MV concept he begins to notice viruses everywhere, and he is equally adept at naming them. Invariably, at the mere utterance of the words "Bourbaki virus," a knowing smile breaks across the mathematician's face, as if he is sure that we are both privy to some private indecency. I will not attempt to describe the Bourbaki virus, for I doubt that the field of MV diagnostics is sufficiently mature for the task. It is an important task, nonetheless, in view of evidence that mathematics is still suffering from the ravages of a "Bourbaki epidemic."

While it may be suspected that many MVs are at large, in any attempts to detect and neutralize them we must be alert to the dangers of misdiagnosis. One man's purported MV may be another man's inspiration. Medical history is riddled with mistakes in diagnosis and treatment. To avoid similar mistakes we must carefully establish suitable diagnostic criteria; we cannot rely on mere hunches or opinions. The main burden of this paper is to set down some specific criteria for the general mental health of mathematics and use these criteria to identify several MVs by their deleterious effects.

2. THE BIOHISTORY OF MATHEMATICS.

There is much to be learned from comparing the evolution of mathematics with the evolution of living organisms. The parallelism is striking enough to suggest that they are governed by common evolutionary laws, including laws of growth and development, variation, adaptation, and competitive selection. Might this be because mathematicians themselves are subject to the laws of biology? In both the organic and the mathematical domains the evolution has been from simple to complex systems. At every evolutionary stage the systems have a high degree of integrity. However, Hilbert's famous assertion that mathematics is an "indissoluble whole," is more the expression of an ideal than a matter of fact. In actuality, there is not one alone but many extant mathematical systems, although it might be said that they have inherited a common genetic structure from the real number system. Each system is well adapted to a particular niche in the great world of pure and applied mathematics. However, alternative systems often lay claim to the same territory, so in the ensuing competition among them the fittest may be the sole survivor.

Crossbreeding of mathematical systems is common, but the results are not always salutary. For example, more than a century ago P. G. Tait opined that the vector calculus of Gibbs is a "hermaphrodite monster" born of Hamilton's quaternions and Grassmann's algebra of extension. Sometimes, however, the offspring of crossbreeding is superior to both parents in versatility and adaptability. So it is with Clifford algebra, born of the same parents as vector calculus.

The evolution of the various mathematical systems can be traced to a common ancestry in the primal integers. There is evidence that Clifford algebra lies on the main branch of the evolutionary tree (Chap. I of Ref. [I]), though there are many other branches and some return by crossbreeding to reinvigorate the main branch. The recent emergence of genetic engineering promises to transform evolutionary development from a matter of chance to a science. I myself have been engaged in a mathematical eugenics of this sort, interbreeding Clifford algebra with tensor calculus, differential forms and many other mathematical systems. The objective of this activity is no less than to approach the Hilbertian ideal of a mathematical supersystem surpassing in power and versatility all of its predecessors. The end product is called *Geometric Calculus* because it is a unified mathematical language for expressing the full range of geometrical properties, relations and structures.

Like the biological world, the world of mathematical ideas is fraught with peril. Every mathematical idea and system must compete for the available territory. Victory of new, superior ideas is by no means assured, for entrenched ideas have a territorial advantage which is sometimes sufficient compensation for inferiority. Besides, the more complex the system the longer it must be nurtured till it is mature enough to hold its own. In particular, the system must be protected from viral attacks which can produce dysfunction and genetic damage. Mathematical viruses are particularly insidious, because they often appear to be harmless or even attractive ideas on their own. Indeed, a virus is most likely to be a remnant of an earlier evolutionary stage in which it had a useful functional role to play; a role made defunct by subsequent evolution. My concern below is to identify specific viruses which attack *Geometric Calculus* and so are inimical to the unification of mathematics.

3. THE UNIFICATION OF GEOMETRY WITH ALGEBRA

The intertwined evolution of geometry and algebra is a long and complex story [1]. Here we focus on their ultimate unification in an integrated conceptual system. This has been taking place over the last century in three stages; from Clifford Algebra to Geometric Algebra to Geometric Calculus ([2], [3]). We shall identify several prevalent viruses which actively retard this process.

Geometric Algebra (GA) is Clifford Algebra (CA) with a geometric interpretation. The interpretation makes GA incomparably richer than CA by itself, leading to a complex network of mathematical structures and theorems as well as applications to every branch of physics. CA is no more than a generalized arithmetic, whereas GA turns it into the grammar of geometric structure. All this has been expounded at length before ([3], [2], [1], [4], [5]). However, appreciation of its significance has been retarded by a virus which is rampant in the literature on Clifford Algebra and consequently infects even specialists in the field. I call it the *quadratic form virus* and characterize it by

MV/Q: Clifford Algebra is the algebra of a quadratic form.

At first sight this proposition appears to be completely innocuous and even helpful. It is true that to every quadratic form there corresponds a Clifford Algebra which is useful for characterizing its properties. The insidious thing about MV/Q is that the verb has often been tacitly interpreted as an "exclusive is," read: CA *is nothing but* the algebra of a quadratic form! This has helped confine CA to a minor subspecialty with no hint of its potential as a grand unifying nexus for the whole of mathematics.

A weaker strain of the MV/Q virus contaminates the minds of many who understand CA as a geometric algebra. From the connection of CA with quadratic forms, they conclude that CA is relevant to metrical geometry alone. The demonstration in [5] that the full structure of CA is essential to the characterization of nonmetrical geometry should be a sufficient antidote for this viral strain. As an inoculation against MV/Q and related viruses, let me summarize how GA unifies and generalizes two of the most basic structures in mathematics and physics.

Geometric algebra is subject to not just one but many different geometric interpretations. This has the advantage of unifying diverse geometric systems by revealing that they share a common algebraic substructure. I call the two most important interpretations the *metrical* and the *projective* interpretations for reasons made obvious by the following example. Let $\mathcal{R}_3$ denote the 8-dimensional Clifford Algebra generated by the real 3-dimensional vector space $\mathcal{R}^3$. I assume that the reader is familiar with Clifford Algebra, and I adopt the notation recommended in [3]. By the way, the practice of defining Clifford Algebras in terms of a basis is almost universal in the literature, a measure of how pervasively the coordinate virus MV/C has infested the field. Although the practice is not wrong and is even advantageous for certain purposes, it introduces irrelevancies which detract from invariant geometric interpretations, and greatly complicate many applications.

The algebra $\mathcal{R}_3$ is by far the most useful geometric algebra in physics because $\mathcal{R}^3$

can be interpreted as a model of the 3-dimensional Euclidean space of experience. This induces a *metrical interpretation* for all $\mathcal{R}_3$. The "geometric product" $\mathbf{ab}$ of two vectors can be decomposed into a symmetric product $\mathbf{a}\cdot\mathbf{b}$ and an antisymmetric product $\mathbf{a}\wedge\mathbf{b}$, as expressed by:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}. \tag{1}$$

Of course $\mathbf{a}\cdot\mathbf{b}$ is the *Euclidean inner product,* while $\mathbf{a}\wedge\mathbf{b}$ is the *Grassmann outer product.* It is important to note that these auxiliary products, besides being of great interest in themselves, perform a critical function that makes it possible to carry out calculations without introducing a basis: they automatically separate elements according to grade. Thus, equation (1) separates the product of vectors (of grade 1) into a scalar part (of grade 0) and a bivector part (of grade 2). More generally, the k-vector part of the geometric product of k-vectors can be identified with the k-fold outer product of Grassmann, as expressed by:

$$<\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_k>_k \;= \mathbf{a}_1\wedge\mathbf{a}_2\wedge\ldots\wedge\mathbf{a}_k. \tag{2}$$

This vanishes if and only if the vectors are linearly independent. In $\mathcal{R}_3$,

$$\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3 = \lambda i, \tag{3}$$

where i is the unit pseudoscalar (3-vector) and $|\lambda|$ is the volume of the parallelepiped determined by the three vectors. Gibbs' vector *cross product* can be defined in $\mathcal{R}_3$ as the dual of the outer product; thus

$$\mathbf{a}\times\mathbf{b} = -i(\mathbf{a}\wedge\mathbf{b}). \tag{4}$$

Since $i^2 = -1$ and i commutes with all vectors, this can inserted in (1) to get

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + i\,\mathbf{a}\times\mathbf{b}. \tag{5}$$

Thus we have the entire vector calculus of Gibbs imbedded in $\mathcal{R}_3$ with *no abuses of standard notation.* The last phrase is italicized to emphasize the important practical fact that, in consequence, $\mathcal{R}_3$ articulates perfectly with the physics literature. Moreover, $\mathcal{R}_3$ generalizes the vector calculus and integrates it perfectly with the quaternion theory of rotations. All this is expounded and exploited at great length in [1]. These brief remarks serve only to illustrate the great unifying power of geometric algebra and to set the stage, by contrast, for an entirely different interpretation and application of $\mathcal{R}_3$.

The *projective interpretation* of $\mathcal{R}_3$ is set by identifying the nonzero vectors in $\mathcal{R}_3$ with points in the projective plane $\mathcal{P}^2$. Two points $\mathbf{a},\mathbf{b}$ are "the same" if and only if $\mathbf{a}\wedge\mathbf{b} = 0$. The entire algebra $\mathcal{R}_3$ now serves to characterize the geometry of the projective plane. The *join* of two distinct points $\mathbf{a}$, $\mathbf{b}$ determines a line $\mathbf{A}$, and this can be expressed by the equation

$$\mathbf{A} = \mathbf{a}\wedge\mathbf{b}. \tag{6}$$

Thus, lines in projective geometry are represented by bivectors. Contrast this with the metrical interpretation of bivectors as "directed areas" or "plane segments." These different interpretations are reflected in differences in the diagrams for the projective and metrical geometries, though our algebraic representation reveals a common underlying algebraic structure.

As indicated by (6), the projective relation 'join' is represented by the outer product. Thus, the join of three distinct points, or the join of a line and a point, is a plane, as expressed by

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = \mathbf{A}\wedge\mathbf{c} = \mathbf{P}. \tag{7}$$

Another basic relation in projective geometry is the meet. The *meet* (or intersection) of lines $\mathbf{A}$ and $\mathbf{B}$ in a point $\mathbf{p}$ can be expressed by the equation

$$\mathbf{A}\vee\mathbf{B} = \mathbf{p}. \tag{8}$$

To see how this relates to the geometric productwe need the concept of duality. Remarkably, duality in geometric algebra is homomorphic to duality in projective geometry. The algebraic dual $\tilde{\mathbf{a}}$ of a vector $\mathbf{a}$ in $\mathcal{R}^3$ is a bivector $\mathbf{A}$ determined by the equation

$$\mathbf{A} = \tilde{\mathbf{a}} \equiv \mathbf{a}i, \tag{9}$$

where, as before, i is the unit pseudoscalar. Now the meet relation in (8) can be defined by writing

$$\mathbf{A}\vee\mathbf{B} = \tilde{\mathbf{a}}\cdot\mathbf{B} = (i\mathbf{a})\cdot\mathbf{B}. \tag{10}$$

Thus, the meet is a composite of duality and the inner product. Despite the appearance of the inner product in (10), the meet is a nonmetrical relation. The meet is independent of signature because the inner product is actually used twice in $(i\mathbf{a})\cdot\mathbf{B} = (i\cdot\mathbf{a})\cdot\mathbf{B}$.

With the projective interpretation just set forth, all the theorems of projective geometry can be formulated and proved in the language of geometric algebra [5]. The theorems take the form of algebraic identities. These identities also have metrical interpretations and therefore potential applications to physics. Indeed, the common formulation in terms of $\mathcal{R}_3$ shows that metrical and projective geometries share a common algebraic structure, the main difference being that projective geometry (at least of the elementary type considered here), employs only the multiplicative structure of geometric algebra. Since both inner and outer products are needed for projective geometry, the geometric product which underlies them is necessary and sufficient as well. This should destroy viruses that would limit the application of geometric algebra to metrical geometry.

4. OPTIMAL STRUCTURE.

A critical question about the organization of mathematics is whether there is an optimal structure for the basic mathematical formalism used to express and develop mathematical ideas. It is now well established that geometric algebra has a greater range of applicability than any other single mathematical system ([1] to [5]), extending to most if not all mathematics amenable to geometric interpretation. Accordingly, we should address the question of whether or not geometric algebra has the optimal structure for expressing geometric ideas. This brings to light a number of mathematical viruses which need to be eradicated.

One ideal of mathematics is generality. Mathematicians are continually striving to prove theorems with broad applicability under the weakest possible assumptions. Very good! However, many mathematicians seem to think that this supports the virulent misconception that weak mathematical structures are more fundamental than strong ones. Among the common viral strains of this misplaced generality we find

MV/G: Grassmann Algebra is more fundamental than Clifford Algebra.

I call this the *Grassmann Virus*, because it is a distortion of Grassmann's own view. That the structure of Grassmann Algebra is contained in Clifford Algebra has been made explicit by defining the outer product as above. Some mathematicians regard this as a mapping of Grassmann Algebra into Clifford Algebra and insist on regarding Grassmann Algebra as a separate entity. Evidently this violates Occam's stricture against unnecessary assumptions, unless it can be given stronger justification than the historical accident that Grassmann Algebra has been developed independently of Clifford Algebra.

Apart from historical accident, the main justification for keeping Grassmann Algebra separate from Clifford Algebra is the mistaken belief that the latter presumes a metric. It is argued that Grassmann Algebra has just the right structure for the theory of determinants, which is nonmetrical. Herein lies another mistake. As commonly understood today, Grassmann Algebra involves only the outer product along with scalar multiplication and vector addition. However, the theory of determinants requires another kind of multiplication which was introduced and employed by Grassmann himself from the beginning. I refer to Grassmann's "regressive product"' which is actually the same as the "meet" defined earlier. From our previous considerations it follows that, since the meet and join (or, equivalently, the inner and outer products) are inherent in the theory of determinants, the entire structure of Clifford algebra is involved as well.

An extensive treatment of determinants in terms of geometric algebra is given in [2], but let us see explicitly why both inner and outer products are needed in the simplest application, namely, *Cramer's Rule in Two Dimensions.* The system of linear equations (over the reals)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= c_1 \\ a_{21}x_1 + a_{22}x_2 &= c_2 \end{aligned} \tag{11}$$

can be expressed as a vector equation in $\mathcal{R}_2$,

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 = \mathbf{c} \tag{12}$$

by selecting any basis vectors $\mathbf{e}^1$, $\mathbf{e}^2$. Equation (11) can be recovered from (12) by introducing a reciprocal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ satisfying $\mathbf{e}^i \cdot \mathbf{e}_k = \delta^i_k$, so that

$$a_{ij} = \mathbf{e}_i \cdot \mathbf{a}_j, \qquad\qquad c_i = \mathbf{e}_i \cdot \mathbf{c}. \tag{13}$$

To solve (12) for the coefficients, say x_1, we use the outer product to obtain

$$x_1 \mathbf{a}_1 \wedge \mathbf{a}_2 = \mathbf{c} \wedge \mathbf{a}_2,$$

whence

$$x_1 = \frac{\mathbf{c} \wedge \mathbf{a}_2}{\mathbf{a}_1 \wedge \mathbf{a}_2}. \tag{14}$$

Note that geometric algebra is needed to express the solution in this simple form, because division is defined therein and the inner product is tacitly employed. Note also that this solution of (12) does not employ a basis. A basis is needed only to relate (12) to (11). To put the solution (14) in the conventional determinant form, we can expand the vectors in a basis to get

$$x_1 = \frac{(c_1 a_{22} - c_2 a_{12})\, \mathbf{e}^1 \wedge \mathbf{e}^2}{(a_{11} a_{22} - a_{21} a_{12})\, \mathbf{e}^1 \wedge \mathbf{e}^2} = \frac{(c_1 a_{22} - c_2 a_{12})}{(a_{11} a_{22} - a_{21} a_{12})}, \tag{15}$$

where the common bivector factors cancel. Conventional coordinate notation disguises the fact that *the inner product is implicit in the use of coordinates,* not just in the sum over pairs of indices, but also in selecting components of a vector, as shown explicitly in (13). A failure to recognize this fact (the coordinate virus again!) accounts for the mistaken belief that only the outer product is involved in determinants. Grassmann knew better ([6],[7])!

A better way to express (14) in terms of determinants, using the inner product explicitly, is

$$x_1 = \frac{\mathbf{c} \wedge \mathbf{a}_2}{\mathbf{a}_1 \wedge \mathbf{a}_2} \frac{\mathbf{e}_2 \wedge \mathbf{e}_1}{\mathbf{e}_2 \wedge \mathbf{e}_1} = \frac{(\mathbf{c} \wedge \mathbf{a}_2) \cdot (\mathbf{e}_2 \wedge \mathbf{e}_1)}{(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{e}_2 \wedge \mathbf{e}_1)}. \tag{16}$$

This employs the most fundamental definition of a k-dimensional determinant as the inner product of two k-vectors. The form on the right side of (15) can be obtained by using the following algebraic identity to expand the determinant

$$(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{e}_2 \wedge \mathbf{e}_1) = \mathbf{a}_1 \cdot \mathbf{e}_1 (\mathbf{a}_2 \cdot \mathbf{e}_2) - \mathbf{a}_1 \cdot \mathbf{e}_2 (\mathbf{a}_2 \cdot \mathbf{e}_1). \tag{17}$$

All this generalizes easily. Thus, the equation

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n = \mathbf{c} \tag{18}$$

in $\mathcal{R}^n$, has the solutions

$$x_k = \frac{\mathbf{a}_1 \wedge \ldots \wedge (\mathbf{c})_k \wedge \ldots \wedge \mathbf{a}_n}{\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n} \tag{19}$$

if the $\mathbf{a}_k$ are linearly independent. And this can be cast in the determinant form

$$x_k = \frac{(\mathbf{a}_1 \wedge \ldots \wedge (\mathbf{c})_k \wedge \ldots \wedge \mathbf{a}_n) \cdot (\mathbf{a}_n \wedge \ldots \wedge \mathbf{a}_1)}{|\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n|^2}, \tag{20}$$

which amounts to using the $\mathbf{a}_k$ themselves as a basis. Finally, the determinants in (20) can be expanded into inner products among vectors (determinants of degree 1) by applying a generalization of (17) derived in [2]. However, there are problems where (20) can be evaluated directly without such an expansion. For such problems the coordinate-free formulation given here is obviously superior to the usual coordinate-based formulation. Clearly, it is superior overall in flexibility of method; coordinates can be employed or not, according to the requirements of the given problem.

It is worth emphasizing once again that the inner product in (16) and (20) need not be given a metrical interpretation, but can be regarded as a device for selecting components of a vector relative to a basis. In other words, *the inner product can function as a linear form.* As noted before, it can do this in a metrically independent way by combining with duality. That was tacitly done in the example above by introducing the reciprocal basis $\{\mathbf{e}^k\}$, which is constructed from a given basis $\{\mathbf{e}_k\}$ by employing duality. To make this explicit for $\mathcal{R}^n$, construct the base pseudoscalar

$$\mathbf{e} = \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n, \tag{21}$$

and the pseudovectors

$$\mathbf{E}^k = (-1)^{(k+1)}\mathbf{e}_1 \wedge \ldots ()_k \ldots \wedge \mathbf{e}_n \tag{22}$$

where the subscripted parenthesis indicates that the kth vector is missing. Then the reciprocal vectors are the duals of the $\mathbf{E}^k$ given by

$$\mathbf{e}^k = \widetilde{E}^k = \mathbf{E}^k\mathbf{e}^{-1}. \tag{23}$$

It is readily verified that

$$\mathbf{e}_j \cdot \mathbf{e}^k = (e_j \wedge \mathbf{E}^k)\mathbf{e}^{-1} = \delta_j^k \tag{24}$$

where it is seen that duality maps the inner products into the outer product to give a result which is manifestly independent of any chosen metric.

The construction of the reciprocal vectors in (23) is of great value in computations with basis vectors. However, its main interest here is that, by employing the inner product and duality, it establishes an isomorphism between the so-called "dual space" of linear forms on $\mathcal{R}^n$ and the "algebraic dual space" of pseudovectors in $\mathcal{R}_n$. This should be a sufficient antidote for the widespread *dual space virus:*

MV/DS: The dual space of forms is more general than an inner product on a vector space.

This virus has been detected by others, including Gian-Carlo Rota and his coworkers [7]. I quote from their extensive critique, in which their concept of "meet" is the same as the one presented and related to the inner product above:

> With the rise of functional analysis, another dogma was making headway namely, the distinction between a vector space V and its dual V^*, and the pairing of the two viewed as a bilinear form.. . .Grassmann's idea was to develop a calculus for the join and meet of linear varieties in projective space, a calculus that is actually realized by the progressive and regressive products. It has been amply demonstrated that this calculus furnishes the definitive notation for such computations. It would, however, be capricious to limit such a calculus to a single operation, just as capricious as limiting the algebra of sets to the single operation of union. Furthermore, the dual space V^* of a vector space V plays no role in such a calculus: a hyperplane is an object living in V, and its identification with a linear functional is a step backwards in clarity. What is needed is an extension of the exterior algebra of V which introduces the second operation — the meet, as we call it — without appealing to vector space duality, and in a notation that is likely to provide the utmost transparency in simultaneously computing with both operations.

Eradication of the MV/G and MV/DS viruses will be a much belated vindication of Grassmann [6], who had the right idea in the first place. These viruses have helped perpetuate the isolated use of weak mathematical structures and so impede the unification of mathematics in the richer structure of Geometric Algebra.

Now we turn from misguided limitations on algebraic structure to equally misguided generalizations of Clifford Algebra. First on the list is the *complexification virus:*

MV/K: Complex Clifford algebras are more general than real Clifford algebras.

This statement is actually false, because every complex Clifford Algebra is contained in a real Clifford Algebra of higher dimension. Nevertheless, the prevalence of complex algebras in the literature points to widespread infection with MV/C. Since complexification of Clifford Algebra does not actually produce greater generality, it has little to recommend it besides custom. Indeed, it has the serious drawback of obscuring geometric meaning, for real scalars have a clear geometric interpretation while complex scalars do not. Moreover, real Clifford algebras contain many square roots of minus one with physical as well as geometrical meaning (See [1], [2] and especially [8]). This suggests that the standard mathematical practice of regarding complex numbers as scalars is an egregious case of mistaken identity, a mistake which can be corrected by recognizing the geometric primacy of the real geometric algebra.

A variant of the complexification virus is found in attempts to use complex scalars to generalize the geometric product. It has been suggested that the standard anticommutation rule for distinct elements of a basis generating a Clifford algebra be generalized to

$$\mathbf{e}_i\mathbf{e}_j = \omega\mathbf{e}_j\mathbf{e}_i, \tag{25}$$

where ω is an nth root of unity. It has recently been shown, however, that the result can always be identified with a standard Clifford Algebra [9], though the basis elements obeying (25) are, of course, not generally vectors in the algebra. Besides, the absence of a geometric interpretation for (25) makes it of dubious value from the standpoint of geometric algebra.

Another product of excessive generalization without due regard to geometric meaning is the *Tensor Virus:*

MV/T: Clifford Algebra can be defined as an ideal in Tensor Algebra.

This statement is true, and in recent decades it has become increasingly popular among mathematicians to define Clifford Algebra in this way. The tacit assumption is that tensor algebra is somehow more fundamental than Clifford Algebra. However, this whole approach suffers from ignoring crucial geometric distinctions, which can be imposed only as an afterthought instead of built in at the beginning. In particular, the completely different geometric roles of symmetric and antisymmetric tensors is ignored. Consequently, the abstract tensor approach suffers from the "folding ruler syndrome" mentioned earlier: it must be "unfolded" before it can be used. Thus, the Clifford Algebra must be factored out of the tensor algebra. However, this factorization plays no role whatsoever in any subsequent applications of Clifford Algebra, and it is obviously a pedagogical impediment to elementary treatments of the subject; not to mention the fact noted by Marcel Riesz [10] that a rigorous factorization raises a difficult question which no one seems to have answered. A surgical excision with Occam's razor followed by a geometrical reconstruction is needed!

No question about the foundations of mathematics is more important than "which axioms should we choose as fundamental?" Here we have the specific problem of choosing between Tensor Algebra and Clifford Algebra. Should we begin with Tensors and derive Clifford Algebra therefrom by factoring? Or should we begin with Clifford Algebra and introduce Tensors as multilinear functions on the algebra (as done in [2])? From the conventional algebraic standpoint the tensor choice is reasonable. However, the geometric considerations which turn Clifford Algebra into geometric algebra raise a deeper question that leads to the opposite conclusion. The deeper question is "What should we choose as our fundamental number system?" As argued in [3] (with further supporting details in [1]), for geometrical purposes the appropriate number system is a real Clifford Algebra, because it provides the optimal representation and arithmetic of the geometric primitives: magnitude direction, orientation and dimension. On the other hand, Tensor Algebra is based on a more limited concept of number embracing only addition and scalar multiplication. Therefore, to imbed the full complement of geometric primitives in the axioms, the roles of Clifford Algebra and Tensor Algebra must be sharply distinguished. The axioms of Clifford Algebra define the basic number system, while Tensors are multilinear functions of those numbers. Tensors are clearly of secondary importance. Numbers before functions!

It is worth noting that when geometric algebra is taken as primitive, the apparatus of tensor products is hardly worth retaining. By way of illustration, consider a tensor T of rank two. Its representation in terms of the tensor product requires the introduction of a basis and has the form

$$T = T^{ij}\mathbf{e}_i \otimes \mathbf{e}_j \tag{26}$$

(summation convention in force!). The quantity T is indeed independent of the chosen basis, but the $\mathbf{e}_i \otimes \mathbf{e}_j$ characterize no more than its rank. Alternatively, geometric algebra defines a reciprocal basis $\{\mathbf{e}^i\}$, so the coefficients can be expressed as a bilinear function on this basis:

$$T^{ij} = T(\mathbf{e}^i, \mathbf{e}^j). \tag{27}$$

But this determines a bilinear function of any vector variables:

$$T(\mathbf{a}, \mathbf{b}) = T^{ij} a_i b_j \tag{28}$$

where $a_i = \mathbf{a} \cdot \mathbf{e}_i$, $b_j = \mathbf{b} \cdot \mathbf{e}_j$. More about all this in [2].

5. VECTOR MANIFOLDS

The development of Geometric Algebra into a full-blown Geometric Calculus capable of handling all aspects of differential geometry requires a fusion of differential and integral calculus with the concepts of geometric algebra. This task has been carried out in [2] with considerable detail. However, it involves a reformulation of manifold theory at the most fundamental level, so mathematicians infected with certain viruses of tradition may find it hard to swallow.

The differentiable manifold is rightly recognized as the general arena for differential and integral calculus. Unfortunately, the conventional definition of "manifold" is couched in terms of coordinates, and this makes the calculus coordinate dependent at its foundations, so much fussing about is needed to derive invariant structures. In contrast, Geometric Calculus makes it possible to dispense with coordinates and deal with invariants exclusively. The key idea is to impose algebraic structure on a manifold by regarding the points as vectors obeying the rules of Geometric Algebra for addition and multiplication. This algebraic structure is then used, instead of coordinates, to define the intrinsic structure of the manifold. A manifold defined in this way is called a *vector manifold*, though it is claimed to be essentially equivalent to the conventional definition of manifold. In some quarters, the very suggestion that manifolds might be defined without the elaborate system of charts and atlases in conventional manifold theory provokes spasms of mathematical outrage reminiscent of the initial response among mathematicians to Dirac's delta function. The summary judgement that "this is mathematical nonsense" is too quick and easy in both cases, and it signals the presence of insidious mathematical viruses.

The characterization of vector manifolds in [2], like Dirac's treatment of the delta function, might be criticized as deficient in rigor. However, no pretense to complete mathematical rigor was made in [2]; rather, analytic details which can readily be supplied by a competent mathematician were omitted, so as to focus attention on the unique features of vector manifolds. The task remains for some ambitious young mathematician to produce a fully rigorous treatment of vector manifold theory.

An alternative treatment of differentiation and integration on vector manifolds with much to recommend it is given by Sobczyk in this volume [11]. Sobczyk limits his treatment to manifolds embedded in Euclidean space, but a cursory examination of his paper reveals that his formulation of the basics makes no appeal to any embedding assumption and so is applicable to vector manifolds in all generality.

One of the "knee-jerk" objections to the vector manifold concept is the unsubstantiated claim that it is less general than the conventional manifold concept. Two arguments have been proffered in support of this claim. Both are wrong, and both can be attributed to viral infections which we are already familiar with. The first argument holds that the very use of geometric algebra presumes an embedding of the manifold in a vector space. The mistake implicit in this argument is the tacit assumption that the necessary algebraic structure can only be defined in terms of a basis, which, of course, also determines a vector space. Here again the coordinate virus is at work! Again, the antidote is to realize that addition and multiplication of points can be defined abstractly without assuming closure under addition and scalar multiplication, that is, without assuming that the points generate a vector space of some finite dimension. Thus, embedding spaces are not presumed in vector manifold theory. On the contrary, geometric calculus may be the ideal tool for proving the known embedding theorems and possibly for discovering new ones.

The second mistaken argument against vector manifold theory holds that the theory is limited to metric manifolds, so it is less general than conventional manifold theory. Attentive readers will recognize the quadratic form virus at work here! It is true that Geometric Algebra automatically defines an inner product on the tangent spaces of a vector manifold. But we have seen that this inner product can be interpreted projectively and so need not be regarded as defining a metric. Moreover, our earlier considerations tell us that the inner product cannot be dispensed with, because it is needed to define completely the relations among subspaces in each tangent space. On the other hand, it is a well-known theorem that a Riemannian structure can be defined on any manifold. Possibly this amounts to no more than providing the inner product on a vector manifold with a metrical interpretation, but that remains to be proved.

For modeling the spacetime manifold of physics, vector manifold theory has many advantages over the conventional approach. For the spacetime manifold necessarily has both a pseudoRiemannian and a spin structure. To model these structures the conventional "modern" approach builds up an elaborate edifice of differential forms and fibre bundles [12], whereas vector manifolds generate the structure as needed almost automatically [13]. Moreover, only geometric calculus provides a common mathematical language that smoothly articulates all branches of physics from relativistic and quantum mechanical to nonrelativistic and classical. The vector manifold concept is needed to formulate all this in a coordinate-free way.

Geometric Calculus has been established as an efficient mathematical language in every domain of physics (see references). No other mathematical system has such scope and power. Accordingly, Geometric Calculus can be expected to play a key role in the progressive unification of physics as well as mathematics. Viruses which impede these unification processes must therefore be eradicated for the general good health of mathematics. To that high purpose this paper is dedicated!

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REFERENCES

[1] D. Hestenes (1986), **New Foundations for Classical Mechanics**, D. Reidel Publ. Co., Dordrecht/Boston.

[2] D. Hestenes and G. Sobczyk (1984), **Clifford Algebra to Geometric Calculus**, D. Reidel Publ. Co., Dordrecht/Boston.

[3] D. Hestenes (1986), A Unified Language for Mathematics and Physics. In: J. S. R. Chisholm and A. K. Common (eds.), **Clifford Algebras and their Applications in Mathematical Physics**, D. Reidel Publ. Co., Dordrecht/Boston, pp. 1–23.

[4] D. Hestenes (1966), **SpaceTime Algebra**, Gordon and Breach, New York.

[5] D. Hestenes (1988), Universal Geometric Algebra, *Simon Stevin* **62**, 253–274.

[6] I. Stewart (1986), Herman Grassmann was right, *Nature* **321**, 17.

[7] M. Barnabei, A. Brini and G. C. Rota (1985), On the Exterior Calculus of Invariant Theory, *J. Algebra* **96**, 120–160.

[8] D. Hestenes (1986), Clifford Algebra and the Interpretation of Quantum Mechanics. In: J. S. R. Chisholm and A. K. Common (eds.), **Clifford Algebras and their Applications in Mathematical Physics**, D. Reidel Publ. Co., Dordrecht/Boston, pp. 321–246.

[9] K. R. Greider and T. Weideman, Generalized Clifford Algebras as Special Cases of Standard Clifford Algebras, *J. Math. Phys.* (submitted).

[10] M. Riesz (1958), *CLIFFORD NUMBERS AND SPINORS*, The Institute for Fluid Dynamics and Applied Mathematics, Lecture Series No. 38, University of Maryland, p. 57.

[11] G. Sobczyk (1991), Simplicial Calculus with Geometric Algebra, (these proceedings).

[12] I. M. Benn and R. W. Tucker (1987), **An introduction to spinors and geometry with applications to physics**, A. Hilger, Bristol/Philadelphia.

[13] D. Hestenes (1987), Curvature Calculations with SpaceTime Algebra, *Int. J. Theo. Phys.* **25**, 581–588; Spinor Approach to Gravitational Motion and Precession, *IJTP* **25**, 589–598.